FREQUENCY-DOMAIN ROBUSTNESS ANALYSIS OF A DISCRETE-TIME STATE-DERIVATIVE FEEDBACK CONTROL LAW

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Abstract— This paper investigates the robustness of a discrete-time state-derivative feedback controller compared to an equivalent state feedback controller. For this purpose, the closed-loop system is cast in the form of a unit feedback structure in order to obtain stability margins at the plant input. More specifically, the resulting open-loop transfer functions are employed to evaluate the robustness with respect to the introduction of time delay. A case study involving the suppression of vibrations in a mechanical system is presented for illustration. As a result, the state feedback controller was found to tolerate a larger time delay compared to the state-derivative feedback controller, despite the equivalence in the nominal case. This analysis is useful to motivate further research concerning the design of robust state-derivative feedback controllers for the discrete-time case.

Keywords— state-derivative feedback, discrete-time control, frequency response, robustness to time delay

1 Introduction

State-derivative feedback (SDF) formulations have emerged recently as a useful tool in view of the available measurements in some practical control problems, such as vibration suppression systems (Abdelaziz and Valášek, 2004), active suspension of vehicles (Reithmeier and Leitmann, 2003; da Silva et al., 2013), optimal motion of vehicles (Fallah et al., 2013), vibration control of landing gear components (Kwak et al., 2002), and vibration control of bridge cables (Duan et al., 2005). The main sensor employed in the instrumentation sets of these problems are accelerometers. From the signals measured by the accelerometers it is possible to reconstruct with good precision the velocities, but not the displacements (Abdelaziz and Valášek, 2004). Therefore, it may be more convenient to express the control law in terms of the velocities and accelerations, which correspond to the derivative of the state variables.

Although the problem of SDF design has been devoted to continuous-time controllers. For instance, pole placement for linear systems using SDF (Abdelaziz and Valášek, 2004), linear quadratic regulator (LQR) design (Duan et al., 2005), stabilizability and stability robustness of SDF controllers (Michiels et al., 2008), robust stabilization of descriptor linear systems (Faria et al., 2010), pole placement of multivariable systems (Faria et al., 2009), robust SDF designs based on linear matrix inequalities (LMI) for linear time-invariant systems with model uncertainties (Assunção et al., 2007; Faria et al., 2010; da Silva et al., 2011), and LMI formulations for time delay systems (Jing et al., 2009; Amri et al., 2011).

The issue of discrete-time design of SDF controllers, to the authors’ knowledge, was only discussed in (Cardim et al., 2009) and (Rossi et al., 2013). (Cardim et al., 2009) was mainly concerned with the problem of a suitable analogue controller design followed by conversion to an equivalent digital controller, called digital redesign, but the results also included expressions to obtain the discrete-time SDF controller from a given discrete-time state feedback (SF) con-
controller. In (Rossi et al., 2013), a discrete-time SDF design based on an equivalent SF law was proposed. However, the robustness of the proposed control laws was not analyzed systematically in those papers. Simulations involving random changes in plant parameters were presented in (Rossi et al., 2013), but a robustness analysis employing formal methods was not carried out.

In this context, the present paper investigates the robustness of a discrete-time SDF controller by using frequency domain analysis. Particularly, the robustness to time delay at the plant input is evaluated. For illustration, a simulated case study involving the suppression of vibrations in a mechanical system is employed, as in (Rossi et al., 2013). The results show different robustness properties of the control loop in each case, despite the equivalence between SF and SDF in the nominal case. Such an analysis is of relevance, as it motivates the development of specific methods for the design of robust discrete-time SDF controllers.

This paper is organized as follows. Section 2 reviews the discrete-time SDF design method proposed in (Rossi et al., 2013). Section 3 describes the frequency-domain robustness analysis method, with the SF and SDF cases in Subsection 3.1 and Subsection 3.2, respectively. Specific details concerning the analysis of robustness with respect to time delay are presented in Subsection 3.3. Section 4 introduces the case study and Section 5 presents the results. Finally, concluding remarks are given in Section 6.

2 Discrete-time state-derivative feedback design based on an equivalent state feedback law

The method for discrete-time SDF design proposed in (Rossi et al., 2013) is presented in this section, for convenience of the reader.

Consider a system described by a continuous-time model of the form
\[ \dot{x}(t) = A_c x(t) + B_c u(t), \quad \forall t \in \mathbb{R} \] (1)
where \( x(t) \in \mathbb{R}^{n_x} \) is the state vector, \( u(t) \in \mathbb{R}^{n_u} \) is the control input, and \( A_c \in \mathbb{R}^{n_x \times n_x}, \ B_c \in \mathbb{R}^{n_x \times n_u} \) are constant matrices. The state \( x(t) \) is assumed to be available for feedback at sampling times \( t = kT, \ k \in \mathbb{Z} \), where \( T \) is the sampling period. Moreover, consider that closed-loop control is generated by a linear state-feedback law with a given gain matrix \( L \in \mathbb{R}^{n_u \times n_x} \) as
\[ u(kT)^+ = -Lx(kT) \] (2)
where superscript + is employed to indicate that the control is updated immediately after the state is measured at each sampling time. Finally, assume that a zero-order hold is employed to keep the control \( u(t) \) constant between sampling times:
\[ u(t) = u(kT)^+, \ (kT)^+ \leq t \leq (k+1)T \] (3)
as illustrated in Fig. 1.

The goal of the discrete-time SDF controller design proposed in (Rossi et al., 2013) consists of matching the control actions provided by the state feedback control law (2). The resulting controller is presented in Theorem 1.

**Theorem 1.** Assume that matrix \( A_c \) is invertible and that the state derivative \( \dot{x}(t) \) is available for feedback at each sampling time \( t = kT \). If the control law given by
\[ u(kT)^+ = -LA_c^{-1}[\dot{x}(kT) - B_c u((k-1)T)^+] \] (4)
is employed with system (1), then the control values will match those provided by the state feedback law (2).

**Proof.** Under the assumption that the control is only updated after the sensor readings are acquired, the state derivative of system (1) at time \( t = kT \) is given by
\[ \dot{x}(kT) = A_c x(kT) + B_c u((k-1)T)^+ \] (5)
Assuming that matrix \( A_c \) is invertible, it follows from (5) that
\[ x(kT) = A_c^{-1}[\dot{x}(kT) - B_c u((k-1)T)^+] \] (6)
By substituting (6) for \( x(kT) \) in (2), it results in the control law (4).

3 Frequency-domain robustness analysis

The present work will be concerned with the particular case of single-input systems (i.e. \( n_u = 1 \)). More specifically, the robustness analysis will be carried out by casting the closed-loop system in the form of a unit feedback structure, as depicted in Fig. 2. In this representation, \( U(z) \) corresponds to the \( Z \)-transform of the discrete-time control sequence \( u(kT)^+ \), whereas \( W(z) \) denotes the \( Z \)-transform of an output sequence \( u(kT)^+ \), which will be conveniently defined in the state feedback and state derivative feedback cases.

By casting the closed-loop system in this form, the transfer function \( G(z) \) can be used to calculate stability margins at the plant input.

Figure 1: Control input \( u(t) \) update at the sampling times (adapted from (Rossi et al., 2013)).

Figure 2: Unit feedback block diagram.
3.1 State feedback (SF)

Fig. 3 shows a block diagram representation of the closed-loop system resulting from the use of the SF control law (2). Digital-to-analog and analog-to-digital operations are indicated by the D/A and A/D blocks, respectively. The output sequence is

$$w(kT)^+ = Lx(kT)$$  \hspace{1cm} (7)

where $L$ is the state feedback gain.

Assuming that the control $u(t)$ remains constant between sampling times, as in (3), the state equation (1) can be discretized as (Franklin et al., 1997)

$$x((k+1)T) = Ax(kT) + Bu(kT)^+$$  \hspace{1cm} (8)

with $A = e^{AT}$ and $B = \int_0^T e^{A\tau}B_\tau \, d\tau$.

Thus, the associated open loop transfer function is given by

$$G_{SF}(z) = L(zI - A)^{-1}B$$  \hspace{1cm} (9)

3.2 State-derivative feedback (SDF)

Fig. 4 shows a block diagram representation of the closed-loop system resulting from the use of the SDF control law (4). A unit sampling time delay is represented by $z^{-1}$. As can be seen in Fig. 4, the output sequence is

$$w(kT)^+ = LA_c^{-1}[\dot{x}(kT) + B_\circ w((k-1)T)^+]$$  \hspace{1cm} (10)

In view of (5) and (10), it follows that

$$w(kT)^+ = Lx(kT) + LA_c^{-1}B_\circ w((k-1)T)^+ + LA_c^{-1}B_\circ w((k-1)T)^+$$  \hspace{1cm} (11)

Therefore, by defining a new state vector as

$$\pi(kT) = [x(kT)^+ w((k-1)T)^+ u((k-1)T)^+]$$  \hspace{1cm} (12)

the output sequence can be written as

$$w(kT)^+ = \mathcal{C}\pi(kT)$$  \hspace{1cm} (13)

with $\mathcal{C} = [L \quad LA_c^{-1}B_\circ \quad LA_c^{-1}B_\circ]$, where $L$ is the state feedback gain.

In the view of the discretized state equation (8), as well the relation between $w(kT)^+$ and $\pi(kT)$ in (13), the open-loop model becomes

$$\pi((k+1)T) = \mathcal{A}\pi(kT) + \mathcal{B}u(kT)^+$$  \hspace{1cm} (14)

with

$$\mathcal{A} = \begin{bmatrix} A & 0 & 0 \\ L & LA_c^{-1}B_\circ & LA_c^{-1}B_\circ \end{bmatrix}, \quad \mathcal{B} = \begin{bmatrix} B \\ 0 \\ 1 \end{bmatrix}$$  \hspace{1cm} (15)

Therefore, the associated open loop transfer function is given by

$$G_{SDF}(z) = \mathcal{C}(zI - \mathcal{A})^{-1}\mathcal{B}$$  \hspace{1cm} (16)

3.3 Robustness with respect to time delay

The robustness analysis which will be carried out in the case study is concerned with time delays that could result, for instance, from communication processes over a digital network. In the presence of a time delay of $d$ sampling periods between the controller and the actuator, the closed-loop system shown in Fig. 2 changes to the form presented in Fig. 5.

$$U(z) \rightarrow z^{-d} \rightarrow G(z) \rightarrow W(z)$$  \hspace{1cm} (17)

Figure 5: Unit feedback block diagram in the presence of time delay.

In this case, the gain and phase of the open-loop system at a frequency $\omega$ are given by (Franklin et al., 1997):

$$|z^{-d}G(z)|_{z=e^{j\omega T}} = |e^{-jd\omega T}| |G(e^{j\omega T})| = |G(e^{j\omega T})|$$  \hspace{1cm} (17)
\[ \begin{array}{c}
    z^{-d}G(z)|_{z=e^{j\omega}} = e^{-jd\omega T} + \int G(e^{j\omega T}) \\
    = -d\omega T + \int G(e^{j\omega T})
\end{array} \] (18)

Since the time delay does not affect the gain values, the closed-loop stability will be retained if \( \omega Td \) is smaller than the phase margin (PM) at the 0 dB crossover frequency \( \omega_c \). In the case of \( N \) crossover frequencies \( \omega_{c,1}, \omega_{c,2}, \ldots, \omega_{c,N} \), this condition becomes

\[ d < \min_{i=1,2,\ldots,N} \frac{PM_i}{\omega_{c,i}T}, \] (19)

where \( PM_1, PM_2, \ldots, PM_N \) are the associated phase margins.

4 Case study

A mechanical system for vibration suppression is presented in Fig. 6 (Abdelaziz and Valašek, 2004). The controlled vibration absorber \( m_2 \) is used to suppress the vibrations of the primary mass \( m_1 \). The plant dynamics are described by the continuous-time state equation (1) with the following matrices \( A_c \) and \( B_c \):

\[ A_c = \begin{bmatrix}
    0 & 0 & 1 & 0 \\
    0 & 0 & 0 & 1
\end{bmatrix}
\] (20)

\[ B_c = \begin{bmatrix}
    \frac{1}{m_2} & \frac{1}{m_2}
\end{bmatrix}' \] (21)

The state vector is given by \( x = [x_1 \ x_2 \ \dot{x}_1 \ \dot{x}_2] \)', where \( x_1 \) and \( x_2 \) denote the vertical displacements of masses \( m_1 \) and \( m_2 \), respectively, and \( \dot{x}_1 \) and \( \dot{x}_2 \) the corresponding velocities. The control input \( u \) is a force provided by an actuator between the two masses. The parameters \( k_1 \) and \( k_2 \) are spring constants, and \( b_1 \) and \( b_2 \) are damping constants.

The open-loop poles for this system (i.e. the eigenvalues of \( A_c \)) are \( \lambda_{1,2} = -2.18 \pm 70.13i \) and \( \lambda_{3,4} = -0.92 \pm 51.30i \). It is assumed that the closed-loop poles should be placed at \( s_{1,2} = -20 \pm 35i \) and \( s_{3,4} = -10 \pm 55i \), as in (Rossi et al., 2013). A discrete-time state-feedback controller was designed by adopting a sampling period \( T = 0.01s \). For this purpose, the desired positions for the closed-loop poles were mapped from the \( s \)-plane according to \( z = e^{sT} \) (Franklin et al., 1997), which resulted in \( z_{1,2} = 0.769 \pm 0.281i \) and \( z_{3,4} = 0.771 \pm 0.473i \).

After discretizing the model as in (8) and using a pole placement procedure with the resulting \((A,B)\) matrices, the gain matrix \( L = 10^4 \times [1.985 - 2.487 - 0.119 \ 0.018] \) was obtained for the discrete-time state feedback controller (2). The same gain matrix \( L \) was used in the state-derivative feedback control law (4).

A small time delay of \( T/100 \) was introduced between sensor readings and control update at each sampling time, to represent the delay associated to A/D conversion, computational processing and D/A conversion.

For the robustness study, an additional time delay was introduced at the system input, as discussed in Section 3.3. In order to determine the maximum delay that still preserves the stability of the system, the frequency responses associated to \( G_{SF}(z) \) and \( G_{SDF}(z) \) up to \( \omega = \pi/T \) were analyzed and the admissible delays were calculated according to (19).

![Mechanical system for vibration suppression](image)

5 Results and discussion

Fig. 7 presents the frequency response obtained from the open-loop transfer function \( G_{SF}(z) \), for the state feedback case. As can be seen, the magnitude crosses 0 dB at \( \omega_{c,1} = 32.1 \) rad/s and \( \omega_{c,2} = 93.2 \) rad/s, which correspond to phases \( \phi_1 = 157.4^\circ \) and \( \phi_2 = -72.3^\circ \), respectively, with phase margins \( PM_1 = 337.4^\circ \) = 5.89 rad and \( PM_2 = 107.7^\circ \) = 1.88 rad. In this case, the stability condition (19) becomes \( d < \min(18.35, 2.02) \). Thus, the maximum integer delay that preserves the stability of the closed-loop system is \( d_{max} = 2 \).

Fig. 8 shows the frequency response obtained from the open-loop transfer function \( G_{SDF}(z) \), for the state-derivative feedback case. As can be observed, the magnitude crosses 0 dB at \( \omega_{c,1} = 35.4 \) rad/s and \( \omega_{c,2} = 132 \) rad/s, which correspond to phases \( \phi_1 = 115^\circ \) and \( \phi_2 = -122^\circ \), respectively, with phase margins \( PM_1 = 295^\circ \) = 5.15 rad and \( PM_2 = 58^\circ \) = 1.01 rad. In this case, the stability condition (19) becomes \( d < \min(14.55, 0.77) \). Thereby, a delay of a single sampling period (\( d = 1 \)) will destabilize the system.

These results were validated by simulating the closed-loop system in the nominal case (no delay, \( d = 0 \)) and in the presence of delay (\( d = 1 \)). Fig. 9 shows the time response of the displacements and
control using the discrete-time state feedback controller without delay. The result obtained by using state-derivative feedback was exactly the same, as expected due to the equivalence between the two control laws established in Section 2.

![Bode diagram of the system using SF.](image1.png)

**Figure 7**: Bode diagram of the system using SF.

![Bode diagram of the system using SDF.](image2.png)

**Figure 8**: Bode diagram of the system using SDF.

![Time response of the displacements and control in the nominal case (no delay).](image3.png)

**Figure 9**: Time response of the displacements and control in the nominal case (no delay), using the discrete-time SF or SDF controllers.

In the presence of delay ($d = 1$), the time responses of the displacements and control using the discrete-time state feedback are presented in Fig. 10 and Fig. 11, respectively. In each figure, an extended view over a longer time frame is presented as an inset. As expected, the closed loop system remains stable, although with an increased settling time.

![Time response of the system using SF.](image4.png)

**Figure 10**: Time response of the displacements using the discrete-time SF controller in the presence of delay.

![Time response of the control using the discrete-time SF controller.](image5.png)

**Figure 11**: Time response of the control using the discrete-time SF controller in the presence of delay.

![Time response of the displacements and control using the discrete-time SDF controller.](image6.png)

**Figure 12**: Time response of the displacements and control using the discrete-time SDF controller in the presence of delay.
6 Conclusion

This paper presented a comparative analysis of discrete-time controllers employing state feedback (SF) and state-derivative feedback (SDF) in terms of robustness with respect to uncertainties at the plant input. Particularly, the robustness to the presence of time delay was analyzed. The robustness analysis was carried out by casting the closed-loop system in the form of a unit feedback structure in the SF and SDF cases. The resulting transfer functions were used to calculate stability margins at the plant input. In the presence of a time delay between the controller and the actuator, the maximum admissible delay that retains the closed-loop stability was calculated for each case based on the stability margins. For illustration, a simulated case study was presented to prove the admissible delays. Despite the equivalence in the nominal case, the results show that the robustness properties of the control loop using SF and SDF can be different. Therefore, it is worth that the controller designer is aware that the SDF design in equivalence to the SF, as proposed in (Rossi et al., 2013), may not preserve its robust properties.

Such analysis can be used as a motivation for future studies concerned with the development of robust discrete-time state-derivative feedback controllers in order to account for model uncertainties. Possible investigations along these lines may include LMI-based formulations.

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