VARIABLE GAIN SUPER-TWISTING CONTROL USING ONLY OUTPUT FEEDBACK

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Abstract— In this paper, an output feedback version of the Variable Gain Super-Twisting Sliding Mode Control is proposed for uncertain plants with relative degree one. This extension is achieved by using first order approximation filters to obtain an upper bound for the norm of unmeasured states. A Lyapunov approach is used to show that the proposed control scheme ensures global stability with exact tracking in finite time. As the control signal is continuous the proposed scheme alleviates the chattering phenomenon. The theoretical results are illustrated by simulations.

Keywords— higher-order sliding modes, global exact tracking, output feedback, uncertain systems

1 Introduction

The sliding mode control (SMC) is a robust non-linear control method that is known to be very effective under uncertainty conditions, including unmodeled dynamics, parameter variation, and external disturbances (Utkin, 1978; Edwards and Spurgeon, 1998). Its main advantage is the so-called invariance property, i.e., once the sliding mode has been achieved, the system becomes insensitive to parameter uncertainties and some disturbance classes. However, the discontinuous nature of the control law results in an undesirable chattering effect, which consists of a high-frequency control switching that can lead to system instability.

The sliding modes concept was generalized in Levant (1993) with the introduction of the higher-order sliding modes (HOSM), which preserve the main advantages of classical SMC and provide even greater accuracy. In this case, the sliding surface is defined based not only on the sliding variable, but also on its derivatives. This new approach results in a control law that is smooth and chattering free (Fridman and Levant, 2002). In particular, we highlight the super-twisting algorithm (STA) based on second-order sliding modes, which was developed in order to avoid the chattering effect in relative degree one systems without requiring information about the sliding variable derivative (Levant, 1998).

Recently in Moreno and Osorio (2008), Dávila et al. (2010), and Gonzalez et al. (2012), a generalization, called Variable Gain Super-Twisting Algorithm (VGSTA), was proposed for the STA. Such generalization consists in, as its name suggests, allow the controller gains to vary with time, improving the robustness and ensuring the rejection of a wider class of disturbances.

The conventional Super Twisting Algorithm is based on the homogeneity principle and hence its convergence can be slow when the initial errors are large. Another important modification of the VGSTA is the introduction of non-homogeneous higher-order terms, which provides for faster convergence with large errors. In some sufficiently small vicinity of the sliding surface, the non-homogeneous higher order terms can be neglected in comparison with the standard homogeneous ones, and thus the asymptotic accuracy and finite-time convergence are preserved. Furthermore, such terms also allow the rejection of disturbances growing together with the sliding variable, which is a necessary feature in control of systems whose linear part is not exactly known. Unfortunately, the VGSTA assumes that the state vector is available for control purposes, restricting its applicability.

In this paper, we propose an output feedback version of the VGSTA for minimum phase uncertain plants with relative degree one. To this end, we use first order approximation filters (FOAF) (Hsu et al., 2003) to generate a norm bound for the unmeasured state so that we can obtain upper bounds for the disturbances that are necessary for
the VGSTA implementation. Simulation results show the robustness and finite-time convergence of the proposed control scheme.

2 Preliminaries

In what follows, all $\kappa$‘s denote positive constants. $\pi(t)$ denotes an exponentially decaying function, i.e., $|\pi(t)| \leq Ke^{-\lambda t}, \forall t$, where $K$ possibly depends on the system initial conditions and $\lambda$ is a (generic) positive constant. $| \cdot |$ stands for the Euclidean norm for vectors, or the induced norm for matrices. Here, Filippov’s definition for the solution of discontinuous differential equations is assumed (Filippov, 1964).

For precise meaning of mixed time domain (state-space) and Laplace transform domain (operator) representations, the following notations are adopted. The output $y$ of a linear time-invariant system with transfer function $H(s)$ and input $u$ is denoted $H(s)u$. Then, the following notation is adopted $y(t) = H(s)u(t)$. Pure convolution operations $h(t) \ast u(t)$, where $h(t)$ is the impulse response of $H(s)$, is denoted by $H(s) \ast u$.

The stability margin of a $p \times m$ rational transfer function matrix $G(s) = C(sI - A)^{-1}B$ is given by

$$\gamma_0 := \min_j \{ -\text{Re}(p_j) \}$$

where $p_j$ is a pole of $G(s)$. The system with transfer matrix $G(s)$ is BIBO stable if and only if $\gamma_0 > 0$.

The stability margin of a real matrix $A$ is given by

$$\lambda_0 := \min_j \{ -\text{Re}(\lambda_j) \}$$

where $\lambda_j$ is an eigenvalue of $A$. If $\lambda_0 > 0$, then $A$ is Hurwitz.

3 Problem Statement

The following uncertain linear time-invariant SISO plant is considered

$$\begin{align*}
\dot{x} &= Ax + B[u + d(x,t)], \\
y &= Cx,
\end{align*}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}$, $y \in \mathbb{R}$, and $d(x,t) \in \mathbb{R}$ is a state dependent uncertain nonlinear disturbance. The following assumptions are made:

(A1) The transfer function $G(s) = C(sI - A)^{-1}B$ is minimum phase, that is, all its zeros are in the open left-half plane (Khalil, 2002, p. 515).

(A2) The system is controllable and observable.

(A3) $G(s)$ has relative degree 1, that is, $\lim_{s \to \infty} sG(s) = k_p = CB \neq 0$.

(A4) The sign of the high-frequency gain ($\text{sign}(k_p)$) is known and supposed positive, for simplicity and without loss of generality, since if $k_p < 0$, one can invert the controller sign and replace $k_p$ by $|k_p|$.

From assumptions (A2) and (A3), and the fact that $CB = k_p \neq 0$, it can be showed that applying the following linear transformation:

$$\begin{bmatrix} \eta \\ y \end{bmatrix} = \begin{bmatrix} B^+ \\ C \end{bmatrix} x,$$

the system (1) can be transformed into the following normal form

$$\begin{align*}
\dot{\eta} &= A_{11} \eta + A_{12} y, \\
\dot{y} &= A_{21} \eta + a_{22} y + k_p (u + d),
\end{align*}$$

where $\eta \in \mathbb{R}^{n-1}$ is an unmeasured state and the zero dynamics given by $\dot{\eta} = A_{11} \eta$ is stable, since $G(s)$ is minimum phase from assumption (A1).

The main objective is to determine a control law $u$ such that $y$ tracks in some finite time a reference signal $y_m$. Then, the tracking error is given by

$$e = y - y_m$$

and its dynamics can be described by

$$\dot{e} = k_p u + f(\eta, e, t),$$

where

$$f(\eta, e, t) = A_{21} \eta + a_{22} e + k_p d(\eta, e, t) + a_{22} y_m - \dot{y}_m,$$

which can be considered as a disturbance, rewritten as

$$f(\eta, e, t) = \underbrace{g_1(\eta, e, t)}_{g_1(\eta, e, t)} + \underbrace{g_2(\eta, e, t)}_{g_2(\eta, e, t)},$$

where $g_1(\eta, e, t) = 0$, if $e = 0$. Thus, it follows that

$$g_1(\eta, e, t) = a_{22} e + k_p \{d(\eta, e, t) - d(\eta, 0, t)\},$$

and

$$g_2(\eta, e, t) = A_{21} \eta + k_p d(\eta, 0, t) + a_{22} y_m - \dot{y}_m.$$

For the purpose of developing a control scheme based on the super-twisting algorithm, some upper bounds for the elements of the disturbance and its derivative are needed. In particular, as $\eta$ is an unmeasured state, its necessary to be available a norm bound for it. Such norm bound can be obtained through a first order approximation filter (FOAF) (Hsu et al., 2003).
4 First Order Approximation Filter – FOAF

In order to obtain a norm bound for $\eta$, we make the following assumption:

(A5) A lower bound $\lambda_1 > 0$ for the stability margin of $A_{11}$ is known.

Lemma 1 Consider system (2) under assumption (A5). Let $\gamma_0$ be the stability margin of the transfer function matrix $H(s) := (sI - A_{11})^{-1}A_{12}$ and $\gamma := \gamma_0 - \delta_0$, where $\delta_0 > 0$ is an arbitrary positive constant. Then $\exists c_1, c_2 > 0$ such that the impulse response $h(t)$ of the system (2) satisfies

$$\|h(t)\| \leq c_1 \exp(-\gamma t), \quad \forall t \geq 0,$$

and the following inequalities hold:

$$\|h(t) * g(t)\| \leq c_1 \exp(-\gamma t) \|g(t)\|$$

$$\|\eta(t)\| \leq c_1 \exp(-\gamma t) \|g(t)\| + c_2 \exp\left[-(\lambda_1 - \delta) t\right] \|\eta(0)\|$$

for all $t \geq 0$.

Proof: see Hsu et al. (2003, Lemma 2). \[\square\]

Considering the result obtained in Lemma 1, it is possible to show that

$$\|\eta(t)\| \leq \hat{\eta}(t) + \pi_\eta(t),$$

where

$$\hat{\eta}(t) := \frac{c_1}{s + \gamma} |g(t)|,$$

(6)

with $c_1, \gamma > 0$ being appropriate constants that can be computed by the optimization methods described in Cunha et al. (2008). The exponentially decaying term $\pi_\eta$ accounts for the system initial conditions.

5 Output Feedback Variable Gains Super-Twisting Algorithm

Consider the Variable Gain Super-Twisting Algorithm (VGSTA) proposed in Dávila et al. (2010) and Gonzalez et al. (2012), given by

$$u = -k_1(t, \hat{\eta}, e)\phi_1(e) - \int_0^t k_2(t, \hat{\eta}, e)\phi_2(e)dt,$$

(7)

where

$$\phi_1(e) = |e| \text{sign}(e) + k_3 e,$$

$$\phi_2(e) = \frac{1}{2} \text{sign}(e) + \frac{3}{2} k_3 |e| \text{sign}(e) + k_3^2 e,$$

being this definition a variable gains generalization of the conventional super-twisting algorithm, which is obtained by making $k_1$ and $k_2$ constants, and $k_3 = 0$.

In this paper, in order to propose an output feedback version of the VGSTA considering the presence of the FOAF (6) in the closed-loop system, we make the following assumption regarding the disturbance, which is slightly different from the one made in Dávila et al. (2010) and Gonzalez et al. (2012):

(A6) The disturbance $f(\eta, e, t)$ satisfies the following inequalities:

$$|g_1(\eta, e, t)| \leq \|\eta(t)\| \|\phi_1(e)\|,$$

$$\|\frac{1}{s + \gamma} g_2(\eta, t)\| \leq \|\eta(t)\| \|\phi_2(e)\|,$$

(8)

where $g_1(\eta, e, t) \geq 0$ and $g_2(\eta, e, t) \geq 0$ are known continuous functions.

Considering (7), the closed-loop system can be rewritten as

$$\dot{\eta} = A_{11}\eta + A_{12}(e + y_m),$$

$$\dot{e} = -k_p k_1(t, \hat{\eta}, e)\phi_1(e) + z + g_1(\eta, e, t),$$

$$\dot{z} = -k_p k_2(t, \hat{\eta}, e)\phi_2(e) + \frac{1}{s + \gamma} g_2(\eta, t),$$

(9)

and the main result of the paper is formulated as follows.

Theorem 2 Consider the system (6),(9) under assumptions (A1)–(A6). Suppose that the disturbance satisfies (8) for some known continuous functions $g_1(\eta, e, t) \geq 0$, $g_2(\eta, e, t) \geq 0$. Then for any initial conditions $[\eta_0^T, e_0, z_0]^T$, the sliding surface $\dot{\eta} = e = 0$ will be reached in finite time if the variable gains are selected as

$$k_1(t, \hat{\eta}, e) \geq \delta + \frac{1}{s + \gamma} \left\{ \frac{4}{\beta} \left[ 4e^2 + 2k_p g_1 + g_2 \right]^2 + 2e(g_2 + k_p \beta) + e + g_1(k_p \beta + 4e^2) \right\},$$

$$k_2(t, \hat{\eta}, e) = \beta + 2k_1(t, \hat{\eta}, e),$$

(10)

where $\beta > 0$, $e > 0$ and $\delta > 0$ are arbitrary positive constants.

Proof: see Appendix A. \[\square\]

6 Simulation Results

Consider the following uncertain LTI plant:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} [u + d(x, t)],$$

$$y = \begin{bmatrix} 1 & 1 \end{bmatrix} x,$$

with transfer function

$$G(s) = C(sI - A)^{-1}B = \frac{s + 1}{(s - 1)^2},$$

and high-frequency gain $k_p = CB = 1$. The state-dependent disturbance is given by

$$d(x, t) = |Cx| + 0.5 \sin(8\pi t)$$

$$= |y| + 0.5 \sin(8\pi t).$$
Applying the linear transformation
\[
\begin{bmatrix}
\eta \\
y
\end{bmatrix} = \begin{bmatrix}
B^{-1} & 0 \\
C & 1
\end{bmatrix} x = \begin{bmatrix}
1 & 0 \\
1 & 1
\end{bmatrix} x,
\]
the system can be transformed into the following normal form:
\[
\begin{align*}
\dot{\eta} &= -\eta + y, \\
\dot{y} &= -4\eta + 3y + u + d(x,t),
\end{align*}
\]
and a norm bound for the unmeasured state \(\eta\) can be obtained by means of the FOAF
\[
\dot{\xi}(t) = \frac{2}{s + 0.5}|y(t)|.
\]
The objective is to ensure that the plant output \(y\) tracks in some finite time a reference signal given by
\[
y_m = \sin(2\pi t).
\]
In this case, considering equations (4) and (5), we have
\[
\begin{align*}
g_1(\eta, e, t) &= 3c + |e + y_m| - |y_m|, \\
g_2(\eta, e) &= -4\eta + y_m + 0.5\sin(8\pi t) + 3y_m - \hat{y}_m,
\end{align*}
\]
and
\[
\frac{dg_2}{dt} = 4\eta - 4y + \hat{y}_m \text{sign}(y_m) + 4\pi \cos(8\pi t) + 3\hat{y}_m - \hat{y}_m,
\]
which implies that
\[
\begin{align*}
|g_1| &\leq 3|e| + |e| \\
&\leq \frac{3}{k_p}|\phi_1(e)|, \\
\left|\frac{dg_2}{dt}\right| &\leq 4|\eta| + 4|e + y_m| + 4\pi + 4|\hat{y}_m| + |\hat{y}_m| \\
&\leq 4|\hat{\eta} + \pi_\eta(t)| + 4|e| + 4|y_m| + 12\pi + 4\pi^2 \\
&\leq \max\left\{\frac{2\pi}{k_p}, 8\eta + 8|y_m| + 24\pi + 8\pi^2 + |\pi_\eta(t)|\right\}|\phi_2(e)|,
\end{align*}
\]
where \(|\pi_\eta(t)| = |8\pi_\eta(t)|\), and the upper bounds \(|\hat{y}_m| \leq 2\pi\) and \(|y_m| \leq 4\pi^2\) were considered. Since the system is uncertain only upper and lower bounds for the elements of \(A, B, C\) and \(k_p\) are available for control purposes. Therefore, the VGSTA majorant functions are chosen more conservatively in order to account for the uncertainties in system matrices and high-frequency gain, and given by
\[
g_1(t, \eta, e) = \frac{6}{k_3} \\
g_2(t, \eta, e) = \max\left\{\frac{9}{k_3^2}, 18\eta + 18|y_m| + 51\pi + 8\pi^2\right\}
\]
Then, the VGSTA gains \(k_1\) and \(k_2\) are chosen such that the inequalities (10) hold, as follows:
\[
\begin{align*}
k_1(t, \eta, e) &= \delta + \frac{1}{k_p}\left\{4\epsilon + 2\epsilon \phi_1 + \phi_2^2 + 2\epsilon \phi_2 + \epsilon \phi_1 + \epsilon \phi_2 + 4\epsilon^2\right\}, \\
k_2(t, \eta, e) &= \beta + 2\epsilon k_1(t, \eta, e),
\end{align*}
\]
where \(\delta = 10^{-3}, \epsilon = 1, \beta = 3500,\) and \(k_1 = 0.5\) is a lower bound for \(k_p\). Further, we set \(k_3 = 0.5\).

In this simulation, we chose the plant initial conditions as \(x(0) = [0.5, 0.5]^T\), while the initial conditions of the VGSTA and FOAF integrators were set to zero. The Euler’s method was used with fixed integration step of \(10^{-6}\). The results are presented in Figure 1. For comparison purposes, it is also presented the results obtained by the state-feedback VGSTA, where the FOAF state \(\dot{\xi}\) is replaced by the plant state in the implementation of the VGSTA majorant functions. Though the use of the FOAF may lead to a more conservative control signal, in this case the results obtained by the output-feedback VGSTA are quite similar to the ones obtained by the state-feedback version. Note the finite time convergence of the output tracking error to zero despite the plant uncertainties and the state-dependent disturbance. This result shows the effectiveness and robustness of proposed control strategy. Furthermore, Figure 1 also shows the smoothness of the control signal, which is free of chattering.

![Figure 1: VGSTA performance: output tracking error e, control signal u, and plant output y with (−) output-feedback scheme; (−−) state-feedback scheme. (−) reference signal y_m.](image)

### 7 Conclusion

We conclude in this paper that first order approximation filters can be effectively used to implement a modified Variable Gain Super-Twisting controller based only on output feedback. As in
the state feedback case, the modified controller can also provide chattering alleviation and global stability properties with finite time convergence of the tracking error to zero. The result is obtained considering a similar Lyapunov approach used to establish the stability and robustness properties of the state feedback VGSTA. The proposed control scheme can be applied to uncertain minimum phase plants with relative degree one and can exactly compensate a class of uncertainties/disturbances. Simulation results were presented to validate the theoretical findings and to illustrate the effectiveness of the proposed control strategy.

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**References**


**A Proof of Theorem 2**

*Proof:* Consider the following Lyapunov function candidate:

\[ V(e, z) = \zeta^T P \zeta, \tag{11} \]

where

\[ \zeta = \begin{bmatrix} \phi_1(e) \\ z \end{bmatrix}, \quad P = \begin{bmatrix} (k_p \beta + 4 e^2) & -2 \epsilon \\ -2 \epsilon & 1 \end{bmatrix}, \]

\[ \beta > 0 \quad \text{and} \quad \epsilon > 0. \]

Note that the function \( V(e, z) \) is everywhere continuous, and differentiable everywhere except on the subspace \( S = \{(e, z) \in \mathbb{R}^2 | e = 0\} \).

Now consider the derivative

\[ \dot{\zeta} = \begin{bmatrix} \phi'_1(e) \{ -k_p k_1(t, \dot{\eta}, e) \phi_1(e) + z + g_1(\eta, e, t) \} \\ -k_p k_2(t, \dot{\eta}, e) \phi_2(e) + \frac{2}{\eta} g_2(\eta, t) \end{bmatrix} = \phi'_1(e) A(t, \dot{\eta}, e) \zeta + \gamma(\eta, e, t), \]

where

\[ A(t, \dot{\eta}, e) = \begin{bmatrix} -k_p k_1(t, \dot{\eta}, e) & 1 \\ -k_p k_2(t, \dot{\eta}, e) & 0 \end{bmatrix}, \]

\[ \gamma(\eta, e, t) = \begin{bmatrix} \phi'_1(e) g_1(\eta, e, t) \\ \frac{2}{\eta} g_2(\eta, t) \end{bmatrix}, \]

everywhere in \( \mathbb{R}^2 \setminus S \), where the derivative exists. One can calculate the derivative of \( V(e, z) \) on the same subset as

\[ \dot{V}(e, z) = \zeta^T P \dot{\zeta} + \zeta^T P \dot{\zeta} \]

where

\[ \dot{\zeta} = \phi'_1(e) \{ -k_p k_1(t, \dot{\eta}, e) \phi_1(e) + z + g_1(\eta, e, t) \} \]

\[ + 2 \zeta^T P \gamma(\eta, e, t) \]

\[ - \phi'_1(e) \zeta^T Q(t, \dot{\eta}, e) \zeta + 2 \zeta^T P \gamma(\eta, e, t), \]

\[ Q(t, \dot{\eta}, e) \]

\[ A(t, \dot{\eta}, e) \]

\[ \gamma(\eta, e, t) \]

\[ \dot{\zeta} \]

\[ \dot{\zeta} \]

\[ \dot{\zeta} \]
where
\[ Q(t, \dot{\eta}, e) = -(A(t, \dot{\eta}, e)^T P + PA(t, \dot{\eta}, e)) \]
\[ = \begin{bmatrix} 2\beta_k k_1^2 + 4\varepsilon k_p (2\varepsilon k_1 - k_2), & \varepsilon \\
-4\varepsilon, & 4\varepsilon \end{bmatrix}, \]
\[ \varepsilon \left( k_p k_2 - 2\varepsilon k_1 - \beta \right) - 4\varepsilon^2, \]
\[ \varepsilon \left( k_p k_2 - 2\varepsilon k_1 - \beta \right) - 4\varepsilon^2, \]
being the last equality given considering the gain \[ k_2 \] in (10). Furthermore, from
\[ \zeta^T P \gamma(t, \dot{\eta}, e, \ell) = \{ (k_p + 4\varepsilon^2) \phi_1 - 2\varepsilon z \} \phi_1 g_1 + \{-2\varepsilon \phi_1 + \varepsilon \} \phi_2 g_1, \]
\[ \leq (k_p + 4\varepsilon^2) \phi_1 \| g_1 \| \| g_1 \| + 2\varepsilon \phi_1 \| z \| \| g_1 \|
\[ + \phi_1 \left( (z_1 + |z_1|)(k_p + 4\varepsilon^2) + 2\varepsilon (z_2 + |z_2|) \right) | \phi_1 |^2
\[ + 2\varepsilon (z_1 + |z_1|) \| z \| \| \phi_1 \|, \]
follows that
\[ 2\zeta^T P \gamma \leq \phi_1 \{ \zeta^T \Gamma(t, \dot{\eta}, e, \ell) + \zeta^T \Pi(t) \zeta \}, \]
where
\[ \Gamma(t, \dot{\eta}, e) = \begin{bmatrix} 2\phi_1 (k_p + 4\varepsilon^2) + 4\varepsilon g_2, & \varepsilon \\
\phi_2 + 2\varepsilon g_1, & 0 \end{bmatrix}, \]
\[ \Pi(t) = \begin{bmatrix} 2\phi_1 (k_p + 4\varepsilon^2) + 4\varepsilon |z_2|, & \varepsilon \\
|z_2| + 2\varepsilon |z_1|, & 0 \end{bmatrix}, \]
and \[ \zeta = \begin{bmatrix} |\phi_1| + |z_1| \end{bmatrix}. \]

Considering the inequality \[ \zeta^T Q \zeta \geq \zeta^T Q \zeta, \]
follows that
\[ \dot{V}(e, z) \leq -\phi_1 \{ \zeta^T Q(t, \dot{\eta}, e, \ell) \zeta - \zeta^T \Pi(t) \zeta \}, \]
where
\[ Q(t, \dot{\eta}, e) = Q(t, \dot{\eta}, e) - \Gamma(t, \dot{\eta}, e), \]
and
\[ Q - 2\varepsilon I = \begin{bmatrix} 2\beta_k k_1^2 - 2\phi_1 (k_p + 4\varepsilon^2) - 4\varepsilon (g_2 + k_p \beta) - 2\varepsilon, & \varepsilon \\
-4\varepsilon^2 + 2\varepsilon g_2, & 4\varepsilon \end{bmatrix}. \]

Selecting \[ k_1 \] as in (10), then the matrix \[ Q - 2\varepsilon I \]
is positive definite. Thus
\[ \dot{V}(e, z) \leq -\left( \frac{1}{2|e|^{1/2}} + k_3 \right) \{ 2\varepsilon \| \zeta \| \}^2 - \zeta^T \Pi(t) \zeta, \]
\[ \leq \left( \frac{1}{2|e|^{1/2}} + k_3 \right) \| \phi_1 \| \{ c_1 \| \phi_1 \| + c_2 \| z \| \}
\[ \leq \left( \frac{1}{2|e|^{1/2}} + k_3 \right) \| \phi_1 \| \{ c_1 \| \phi_1 \| + c_2 \| z \| \}
\[ \leq \left( \frac{1}{2|e|^{1/2}} + k_3 \right) \| \phi_1 \| \{ c_1 \| \phi_1 \| + c_2 \| \phi_1 \| \}
\]
\[ \leq c_1 \| \zeta \| + c_2 \| \phi_1 \| \| \phi_1 \|, \]
where
\[ \| \phi_1 \|^2 = |\phi_1|^2 + |z|^2 = |e| + 2k_3 |e|^{3/2} + k_3^2 |e|^2 + |z|^2 \]
is the Euclidean norm of \( \zeta \). It follows from
\[ \lambda_{\min} \{ P \} \| \zeta \|^2 \leq \zeta^T P \zeta \leq \lambda_{\max} \{ P \} \| \zeta \|^2 \]
that
\[ \dot{V}(e, z) \leq e_\ell V^\frac{1}{2}(e, z) + e_2 V(e, z), \]
and \[ V(e, z) \] cannot escape in finite time.

Further,
\[ \zeta^T \Pi(t) \zeta \leq \lambda_{\max} \{ \Pi(t) \} \| \zeta \|^2 \leq |\zeta(t)| \| \zeta \|^2, \]
and equation (12) implies that
\[ \dot{V}(e, z) \leq -\left( 2\varepsilon - |\zeta(t)| \right) \left( \frac{1}{2|e|^{1/2}} + k_3 \right) \| \zeta \|^2. \]
Since \[ |\zeta(t)| \] is an exponentially decaying function, after some finite time \[ t_1 \]
\[ (2\varepsilon - |\zeta(t)|) \geq 2\mu, \quad t \geq t_1, \]
where \( \mu < \varepsilon \) is a positive constant. Then
\[ \dot{V}(e, z) \leq -2\mu \left( \frac{1}{2|e|^{1/2}} + k_3 \right) \| \zeta \|^2, \quad t \geq t_1. \]

From equation (13) and
\[ |e|^{1/2} \leq \| \zeta \| \leq V^\frac{1}{2}(e, z), \]
\[ \lambda_{\min} \{ P \}, \]
follows that
\[ \dot{V} \leq -\kappa_1 V^\frac{1}{2}(e, z) - \kappa_2 V(e, z), \quad t \geq t_1, \]
where
\[ \kappa_1 = \frac{\mu \lambda_{\min} \{ P \}}{\lambda_{\max} \{ P \}}, \quad \kappa_2 = \frac{2\mu k_3}{\lambda_{\max} \{ P \}}. \]

Observing that the trajectories of the VGSTA cannot stay on the subset \( S = \{ (e, z) \in \mathbb{R}^2 | e = 0 \} \), one concludes by Zubov’s stability theorem that the equilibrium point \( (e, z) = 0 \) is reached in finite time for any initial conditions.

Since the solution of
\[ \dot{v} = -\kappa_1 V^\frac{1}{2} - \kappa_2 v, \quad t \geq t_1, \quad v(t_1) = v_{t_1} \geq 0 \]
is given by
\[ v(t) = \left[ v_{t_1} + \frac{\kappa_2}{\kappa_1} \{ 1 - \exp \left( \frac{\kappa_2}{\kappa_1} (t - t_1) \right) \} \right]^{\frac{1}{2}}, \]
it follows that \( (e(t), z(t)) \) converges to zero at most after some finite time
\[ T = t_1 + \frac{2}{\kappa_2} \ln \left( \frac{\kappa_2}{\kappa_1} \sigma_{t_1} (z_{t_1} + 1) \right). \]

\[ \Box \]