REGULABLE CONVERGENCE RADIUS FOR FLEXIBLE AUXILIARY CONTROL LAW IN WHEELED MOBILE ROBOTS SINGULARLY PERTURBED: CURVILINEAR APPROACH

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Abstract—In this paper is proposed a method to reduce the effects of slipping and skidding in WMRs not exactly satisfying kinematic constraints. To this end, it is studied the case for WMRs whose kinematic constraints are violated owing to deformability or flexibility and it is considered the static-state linearization whose robustness will be based on a suitable transformation for curvilinear coordinates with regulable convergence radius. In results will be seen that the tracking error converges to small ball of the origin whose radius can be adjusted by a known function that depends on the slipping and skidding variations.

Keywords—Singualar perturbations method, slipping and skidding variations, wheeled mobile robots, curvilinear coordinates, regulable convergence radius.

1 Introduction

Wheeled mobile robots (WMRs) have two types of kinematic constraints: pure rolling condition and the nonslipping condition, both associated to the contact point between each wheel and the ground, see (Campion et al., 1996). Often, WMRs are based on assumption that these constraints are satisfied at each instant along motion. However, kinematic constraints are violated when a WMR is moving on a trajectory either accelerating, or decelerating, or cornering at a high speed. In addition, several phenomena contribute with that violation, such as sliding, deformability or flexibility. Thus, the trajectory tracking control of WMRs and its stabilization have given rise to an abundant literature in recent years due to its challenging theoretical nature [see (Aithal and Janardhanan, 2013; Song and Boo, 2004; Fernández et al., 2015)]. It is well-known that there does not exist a smooth pure state feedback control law such that the state of a WMR converges to the origin. In order to solve this problem, several types of controllers have been proposed, such as time-varying control laws, discontinuous control laws, and hybrid control laws based on linearization local controlling, nonlinear state feedback with singular parameters, or backstepping ([Aithal and Janardhanan, 2013; D’Andréa-Novel et al., 1995; Le- roquais and D’Andrea-Novel, 1996; Motte and Cam- pion, 2000; Dong, 2010]).

In proposal reported here, we consider the tracking control problem of WMRs which are subject to slipping and skidding effects. For that purpose, it is considered, the classical control law based in static-state feedback linearization whose robustness with respect to the deformability of wheels will be based on singular perturbation methods and an auxiliary control law that is modeled on a suitable transformation for curvilinear coordinates with regulable convergence radius. In results will be seen that the tracking error converges to small ball of the origin whose radius can be adjusted by a known function that depends on the slipping and skidding variations.

2 Theoretical preliminaries on singularly perturbed model for WMRs

Thus, let consider WMRs whose total motion is executed by the action of $N$ wheels such that $N = N_f + N_c + N_o$, where $N_f, N_c, N_o$ represent the number of fixed wheels, centered orientable wheels and off-centered orientable wheels, respectively, see (Campion et al., 1996). Let also consider that the configuration of WMRs is fully described by the vector of generalized coordinates $q = [z \beta_0, \varphi]^T$ (1)

where

$$z = [\xi \beta_0, \varphi]^T \in \mathbb{R}^{3N_o + N_f + N_c}, \quad \xi = [x, y, \theta, \sigma_0, \ldots, \sigma_{N_o-1}]^T,$$

with $3 + N_{ind}$ degrees of freedom, being $N_{ind}$ a value associated with the angular position of the wheels that can be oriented independently. In this paper, we will consider that $N_{ind} = 0$. Furthermore, $\beta_0 \in \mathbb{R}^{N_c}$, $\beta_0 \in \mathbb{R}^{N_f}$ and $\varphi \in \mathbb{R}^{T}$. Each pair $(\beta_i, \varphi_i)$ is associated with guidance and angular pose of $i$-th centered orientable wheel and each pair $(\beta_o, \varphi_o)$ is associated with $i$-th guidance and angular pose of off-centered orientable wheel. The variables $x, y$ represent the pose of the local frame $\{L\}$, related to WMR’s body, with respect to a global frame $\{W\}$ and $\theta$ specifies the orientation. The constant $\delta_s$ is called degree of steerability and it always equal to $N_c$.

The generalized velocity $\dot{q}$ may be written as:

$$\dot{q} = S(q)\eta + A(q)\epsilon \mu \left( \begin{array}{c} \dot{s} \\ \dot{\varphi} \end{array} \right) = \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} \eta + \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \epsilon \mu$$ (2)
where $S(q) \in \mathbb{R}^{(3 + N_u + N_x) \times 4_n}$, $\eta = [u \ z]^T \in \mathbb{R}^4$, and $\mathbf{u} \in \mathbb{R}^{3}$ where $\varepsilon > 0$ is a scale factor related to the flexibility and $A^T(q) \in \mathbb{R}^{K_r \times (3 + N_u + N_x)}$ represents $K_r$ independent constraints, with $K_r < 3 + N_u + N_x$. The constant $\delta_0$ represents the degree of maneuverability. Here, by considering the proposal in (Campion et al., 1996), for the type of wheels previously indicated we will assume that

$$\delta_0 \leq 3$$

and $\delta_0 - \delta_2 \leq 2$. (3)

Now, multiplying both sides of (2) by $A^T(q)$, and due to $A^T(q)S(q) = 0$ we have

$$A^T(q)\dot{q} = A^T(q)A\varepsilon\mu,$$

which represents the kinematic constraints into the singular perturbations approach.

**Assumption 1** $\|A^T(q)A(q)\varepsilon\mu\| < \varepsilon^*$, where $\varepsilon^* > 0$, for $\varepsilon(t)$, is a known function that depends on slipping, skidding and deformation of the wheel.

A singular perturbation model for a WMR can be write as the following space-state, see (Fernández et al., 2015; Fernández et al., 2014b):

$$\begin{aligned}
\dot{x} &= Z_0(q)\eta + [z_1 q_2 q_3]\mu + Z_3(q)\tau, \\
\dot{\mu} &= G_0(q)\eta + [z_1 q_2 q_3]\mu + G_3(q)\tau, \\
\end{aligned}$$

where $x = [q q_T]^T \in \mathbb{R}^{3 + N_u + N_x + N_y}$ is the state vector with slow variables, $\mu \in \mathbb{R}^{K_r}$ is the vector with the fast variables, see (D’Andréa-Novel et al., 1995). The vector fields $Z_0, Z_1, Z_2, Z_3 \in \mathbb{R}^{(3 + N_u + N_x + N_y) \times 4_n}$, $G_0, G_1, G_2, G_3 \in \mathbb{R}^{K_r \times 4_n}$, $G_0 \in \mathbb{R}^{K_r \times N_y}$ are continuously differentiable in the parameters $(x, \mu, \varepsilon, t)$ in $\mathbb{D}_x \times \mathbb{D}_\mu \times [0, \varepsilon_\tau] \times [0, T]$, being $\mathbb{D}_x \times \mathbb{D}_\mu \times [0, \varepsilon_\tau] \times [0, T]$, the vector $\tau \in \mathbb{R}^N$ represents the input torques at motors and usually they are expressed as an uniformly bounded function $\tau = \tau(q, \eta, \mu)$ such that $\tau_{\min} \leq \tau \leq \tau_{\max}$, for $\tau_{\min}, \tau_{\max} \neq 0$. (7)

The system (5)-(6) is a standard singular form if and only if $0 = G_0(q)\eta + G_3(q)\tau$ has $k \geq 1$ different and isolated roots, denoted by:

$$\mu_i = H_i(x, t), i = 1, \ldots, k.$$

For each $i$-th function $\mu_i$, it is defined the following reduced system:

$$\dot{x} = Z_0(q)\eta + Z_3(q)H_i(x, t) + Z_3(q)\tau, \quad \dot{\mu} = 0, \quad x = x_0 \quad \text{corresponding to the case } \varepsilon = 0.$$

**Definition 1** (Fernández et al., 2015) For system (5)-(6) a flexible manifold is defined by the following equation:

$$\mu = H(x, t),$$

thus, for a standard singular form, it is valid that the boundary layer system, in a new variable $\tilde{\mu}$, is defined by

$$\frac{d\tilde{\mu}}{dt} = G_0(q)\eta_0 + G_2(q)\mu_0 + G_3(q)\tau_0,$$

where $\mu_0 = \mu - H(x_0, t_0), x_0 = [q_0 q_T]^T$ are interpreted as fixed parameters, $\tau_0 \geq \tau(q_0, \eta_0, \mu_0)$ is the initial condition and $t = t_0$ has the role of time dilation.

Now, let consider an open ball centered at origin with radius $\bar{r}$ into $D_\delta \subset \mathbb{R}^{3 + N_u + N_x + N_y + 4_n}$, $B_0(0 ; \bar{r})$, and an open ball and centered at origin with radius $\bar{p}$ into $D_\mu \subset \mathbb{R}^{K_r}$, $B_\mu(0 ; \bar{p})$. In addition, let consider the following theorem:

**Theorem 1 (Tikhonov’s theorem)** For a system in a standard form (5)-(6), let consider the following conditions:

i. there exist $T, \bar{r}, \bar{p}$, $r_0 \in \mathbb{R}_+$ such that $Z(x, \mu, \varepsilon, t), G(x, \mu, \varepsilon, t)$ and its partial derivatives with respect to $x, \mu$ and $\varepsilon$ are continuous in $\mathbb{D}_x \times \mathbb{D}_\mu \times [0, \varepsilon] \times [0, T]$; $H(x, t)$ and Jacobian $\partial G(x, \mu, 0, t)/\partial\mu$ have partial derivatives continuous, and, the reduced model (9) has unique solution $\dot{x}$ defined on $[0, T]$ which belongs to $B_0$;

ii. there exists $t^* > 0$ such that $\mu = 0$ is an exponentially stable equilibrium point of the boundary layer system uniformly in the parameters $x_0$ and $t_0$, and, $\mu_0 - \mu(0)$ belongs to its domain of attraction, i.e., $\lim \mu(t^*) = 0$;

iii. the origin of reduced system (9) is exponentially stable.

Then, there exist positive constants $\mu_1 > 0, \nu_2 > 0$ and $\varepsilon^* > 0$ such that if $\|x_0\| < \nu_1, \|\mu_0 - H(x_0, 0)\| < \nu_2$ and $\varepsilon < \varepsilon^*$ then the following approximations are satisfied uniformly with respect to $T$, for $\forall t \in [0, T]$; $T \in \mathbb{R}$:

$$x(t) = \bar{x}(t) + O(\varepsilon) \quad \text{and} \quad \mu(t) = \tilde{\mu}(t) + \mu(t^*) + O(\varepsilon)$$

where $O(\varepsilon)$ represents a quantity in terms of $\varepsilon$.

One way to ensure item (iii) in Theorem 1 is to suppose that the outputs controlled are a subset of the generalized coordinates $q$. Thus, let assume that there exists a partition of the system (5)-(6) in three parts:

$$\begin{aligned}
\dot{z} &= Z_0^1(q)\eta + [z_1 Z_1^1(q) + Z_2^1(q)]\mu + Z_3^1(q)\tau, \\
\dot{w} &= Z_0^2(q)\eta + [z_1 Z_1^2(q) + Z_2^2(q)]\mu + Z_3^2(q)\tau, \\
\dot{\mu} &= G_0(q)\eta + G_1(q)\mu + G_3(q)\tau;
\end{aligned}$$

where $z = [\beta^T \varphi^T]^T \in \mathbb{R}^{3 + N_t}$ is the first parcel, $w = [\beta^T \varphi^T]^T \in \mathbb{R}^{N_t + N\mu}$ is a second parcel and the sums of terms $Z_0^1(q)\eta + [z_1 Z_1^1(q) + Z_2^1(q)]\mu + Z_3^1(q)\tau$ and $Z_0^2(q)\eta + [z_1 Z_1^2(q) + Z_2^2(q)]\mu + Z_3^2(q)\tau$ represent the vector fields $Z_{\text{en}}(z, w, \mu, \varepsilon, t) \in \mathbb{R}^{3 + N_t + N\mu}$ and $Z_{\text{en}}(z, w, \mu, \varepsilon, t) \in \mathbb{R}^{N_t + N\mu}$, respectively, with initial conditions $z(0) = z_0, w(0) = w_0$ and $\mu(0) = \mu_0$.

Whenever $\delta_0$ is full, the range of $S(q, B(q))$ is full and the matrix $S^T(q, B(q))S(q)$ is nonsingular, the generic static state-feedback linearization of (5)-(6) (properly partitioned) is defined by

$$\begin{aligned}
\tau &= [S^T(q, B(q))]^{-1} \left\{ S^T(q, B(q))S(q)v + M(q) \left( \frac{\partial S}{\partial q} S(q) \eta \right) - C(q, S(q)\eta) \right\},
\end{aligned}$$

where $v \in \mathbb{R}^{N_t}$ represents an arbitrary auxiliary control law.

2.1 Auxiliary control law: curvilinear approach

Based on the curvilinear approach (Fernández et al., 2014b) and Fig. 1(a), the tracking control of any point $p$ on a WMR with desired linear velocity $u_1^*$ can be parameterized by

$$q = [q_1 q_2 q_3]^T = \left[ \begin{array}{ccc} \lambda & d & \alpha \end{array} \right]^T = f(z)$$

where $f(z) : \mathbb{R}^{3 + 4_n} \rightarrow \mathbb{R}$ is a known vector field, $\lambda$ is the curvilinear coordinate along any curve, $d$ is the coordinate of point $p$ along $N(\lambda)$ (normal vector at $\lambda$) and $\alpha$ is the WMR’s orientation with respect to $T(\lambda)$ (tangent vector at $\lambda$). In like (Dong, 2010), here will
be assumed that \( \text{curv}[(\lambda)] < 1/R, \forall \lambda \) where \( R > 0 \) is a constant and \( \text{curv}(\cdot) \) represents the curvature.

Let \( e = [e_1, e_2, e_3]^T \) be the tracking error associated with \( q^* \) and \( h \in \mathbb{R}^6 \), the control inputs of the kinematic controller formally defined by

\[
e = \Pi(I)(z) \quad (16)
\]

\[
h = \Pi(h(s)) \eta, \quad \Pi(h(s)) = \text{non-singular matrix (see (Fernández et al., 2014b; Dong, 2010))} \quad \forall t, \quad \text{is a known function used to regulate the convergence radius of the controller.}
\]

From (Fernández et al., 2015) is known that \( v = \dot{\eta} \) thus the auxiliary control law \( v \) is defined by

\[
v = \left( \frac{\partial \Pi_2(z, h_\ast)}{\partial z} (S_I(q) \eta + A_I(q) \dot{\mu}) + \frac{\partial \Pi_2(z, h_\ast)}{\partial h_\ast} h \right) + \Pi_2(z, h_\ast) \dot{\rho} \quad (18)
\]

such that \( h = \rho \in \mathbb{R}^6 \) (from (3), \( \rho \leq 3 \)) with \( u^*_i > \delta_i > 0 \) and

\[
\rho_1 = u^*_i, \\
\rho_2 = \begin{cases} 
-\zeta \dot{\phi}_2 \tanh \left( \frac{e_3 \dot{\phi}_2}{h_\gamma} \right), & \text{if } \delta_i - \delta_d = 2 \\
-\left(k_3 e_3 + e_2 u^*_i \right) u^*_i + \frac{\dot{u}^*_i \phi_1}{(u^*_i)^2}, & \text{if } \delta_i - \delta_d = 1 \\
\zeta^*_1, & \text{if } \delta_i - \delta_d = 2 \\
\zeta^*_2, & \text{if } \delta_i - \delta_d = 1
\end{cases}
\]

\[
\rho_3 = \begin{cases} 
\zeta^*_1, & \text{if } \delta_i - \delta_d = 2 \\
\zeta^*_2, & \text{if } \delta_i - \delta_d = 1 \\
0, & \text{if } \delta_i - \delta_d = 1
\end{cases}
\]

where \( \zeta^*_1, \zeta^*_2 \) are desired quantities, \( k_3 > 0 \) is a constant used like design parameter, \( L \) the abbreviation of Lie Derivative and

\[
\phi_1 = \zeta^* \frac{\partial e_2}{\partial q_2} \tanh \left( \frac{e_3 \dot{\phi}_2}{h_\gamma} \right), \\
\phi_2 = L_{q_1} L_{q_2} e_2 + k_2 L_{\dot{q}_1} e_2 + \frac{\dot{L}_{q_1} \phi_1}{u^*_i}
\]

\[
g_1 = \left[ 1 - \text{curv}(q_1) q_2 \right] T \\
g_2 = \left[ \frac{\sin q_3}{1 - \text{curv}(q_1) q_2} - \cos q_3 - \frac{\text{curv}(q_1) \sin q_3}{1 - \text{curv}(q_1) q_2} \right] T
\]

For more details on outputs of controller, contained in \( \rho \), refer to (Fernández et al., 2014b).

### 3 Slipping and skidding variations

The longitudinal slip for i-th wheel is defined by

\[
s_i = \frac{V_{i,y} - r \dot{\varphi}_i}{\| V_i \|} \quad (19)
\]

where \( r \) is the radius of the wheel, \( V_i \) denote the velocity of the centre of i-th wheel, \( V_{i,y} \) represents the longitudinal component of \( V_i \) and \( \dot{\varphi}_i \) represents the angular velocity. It is important to point out that (19) is a normalized expression, i.e., \( s = +1 \) when \( V_{i,y} < 0 \), \( \dot{\varphi}_i > 0 \) and \( s = -1 \) when \( V_{i,y} > 0 \), \( \dot{\varphi}_i < 0 \). The slip angle \( \delta_{x,i} \) associated with the skidding of i-th wheel is defined as the angle between the wheel plane and the velocity of its centre. Let \( \delta_{x,i} \) characterized by

\[
\delta_{x,i} \approx \sin \delta_{x,i} = \frac{|V_{i,x}|}{\| V_i \|} \quad (20)
\]

where \( V_{i,x} \) is the transversal component of \( V_i \).

**Assumption 2** The velocities of all driving wheels at their center are taken to be identical, and more precisely, equal to their average:

\[
V_i = (\dot{x}_s + \dot{y}_s)^{1/2} = u_1 \quad (21)
\]

where \( u_1 \) is the linear velocity of the WMR.

Based on the proposal in (Motte and Campion, 2000) it will be guaranteed that the transversal and longitudinal forces of i-th wheel are uniformly bounded. Let these forces defined by

\[
F_{i,x} = D \frac{V_{i,x}}{\| V_i \|} \quad \text{and} \quad F_{i,y} = G \frac{V_{i,y} - r \dot{\varphi}_i}{\| V_i \|} \quad (22)
\]

where \( G \) and \( D \) are the slip stiffness and cornering stiffness coefficients, respectively, both depending on the contact nature between wheel and motion surface.

From (Fernández et al., 2015; Fernández et al., 2014a), for motion of a WMR on a trajectory there are \( N \) conditions, corresponding to \( N \) wheels defined by:

\[
\delta_{x,i} = \delta_{x,i} = 0 \quad \text{and} \quad s_i = s_i = 0, \quad (23)
\]

for \( i = 1, \ldots, N \).

#### 3.1 Slipping variation

The slipping variation associated with i-th wheel can be written by differentiating (19) without norms, i.e.,

\[
\dot{s}_i = \frac{V_{i,y} - r \dot{\varphi}_i}{V_i} - \frac{(V_{i,y} - r \dot{\varphi}_i) V_i}{V_i^2} \quad (24)
\]

Assume that the slip angle \( \delta_{x,i} \) is sufficiently small such that \( V_i \approx V_{i,y} \), then

\[
\dot{s}_i = r \dot{\varphi}_i \frac{V_{i,y} - r \dot{\varphi}_i}{V_i} \quad (24)
\]

By using the Euler-Lagrange formulation in (Fernández et al., 2015) for i-th wheel we have

\[
I_r \ddot{\varphi}_i = \tau_i - r F_i \quad (25)
\]

where \( \tau_i \) is the control action applied to i-th wheel, \( F_i \) the i-th traction force (resulting from composition of \( F_{i,x} \) and \( F_{i,y} \)) and \( I_r \) the inertia of the wheel. Substituting (25) in (24) and manipulating for \( N \) wheels yields

\[
\dot{S}_i V_i = (I_N \times N - S_i) V_i - m_i T + r m_i F_i \quad (26)
\]

where \( m_i = r / I_r \), \( T = [\tau_1 \cdots \tau_N]^T \in \mathbb{R}^N \), \( F_i = [F_{i,x} \cdots F_{i,y}]^T \in \mathbb{R}^N \), \( V_i = [V_i(q, \eta, \mu) \cdots V_i(q, \eta, \mu)]^T \in \mathbb{R}^N \) (being \( V_i \neq V_i(q, \eta, \mu) \)), \( S_i = \text{diag} \{s_1, \ldots, s_N\} \in \mathbb{R}^{N \times N} \) and \( I_N \times N \) is a eye matrix with dimension \( N \times N \).

As consequence of the linear approximations in (22) the force \( F_i \) can be expressed as \( F_i = G S_i \dot{\Delta}_i \), being \( \Delta_i = \text{diag} \{\delta_{x,1}, \ldots, \delta_{x,N}\} \in \mathbb{R}^{N \times N} \) the skidding of \( N \) wheels.

**Lemma 1** For WMRs that moving on a trajectory with velocity at center of the wheels \( \| V_i \| \leq V_r \), where \( V_r > 0 \) is a known value, with violation of the appropriate rolling conditions in (23) (i.e., \( \| S_i \| \neq 0 \) and \( \| \dot{\Delta}_i \| \neq 0 \), then it is satisfied that

\[
\gamma_{11} \| \dot{S}_i \|^2 \leq \| V_i \|^2 \leq V_r V_r \gamma_{11} \| \dot{S}_i \|^2 \quad (27)
\]

where \( \gamma_{11}, V_r^2 > 0 \) are known values.

**Proof:** The proof can be seen in (Fernández et al., 2015).
3.2 Skidding variation

The skidding variation \( \delta_{x,i} \) associated with the \( i \)-th wheel can be calculated by differentiating (20) without the norms, i.e.,

\[
\delta_{x,i} = \frac{V_{i,x} V_i - \dot{V}_i V_{i,x}}{V_i},
\]

(28)

The above expression for \( N \) wheels can be rewritten, through algebraic manipulation, as the following compact form

\[
\dot{\Delta}_1 V_i = \dot{V}_i - \Delta_1 \dot{V}_i
\]

(29)

where \( V_i = [V_{i,x}, \ldots, V_{i,N}]^T \in \mathbb{R}^N \).

Lemma 2 For WMRs that moving on a trajectory with velocity at center of the wheels \( ||V_i|| \leq \bar{V}_R \), where \( \bar{V}_R > 0 \) is a known value, with violation of the appropriate rolling conditions in (23) (i.e., \( \|\dot{S}_i\| \neq 0 \) and \( \|\dot{\Delta}_i\| \neq 0 \)), then it is satisfied that

\[
\gamma_{21} \|\dot{\Delta}_1\|^{-1} \leq \|\dot{V}_i\| \leq \bar{V}_R V_{i,2}^{\phi} \gamma_{21} \|\dot{\Delta}_1\|^{-1},
\]

(30)

where \( \gamma_{21} = \bar{V}_R V_{i,2}^{\phi} \gamma_{21} \) and \( \gamma_{22} = \bar{V}_R V_{i,2}^{\phi} \gamma_{21} \).

Proof: The proof can be seen in (Fernández et al., 2015).

\[ \square \]

Theorem 2 ((Fernández et al., 2015)) For WMRs that moving on a trajectory with velocity at center of the wheels \( ||V_i|| \leq \bar{V}_R \), where \( \bar{V}_R > 0 \) is a known value, with violation of the appropriate rolling conditions in (23) (i.e., \( \|\dot{S}_i\| \neq 0 \) and \( \|\dot{\Delta}_i\| \neq 0 \)), then it is satisfied that:

\[
\frac{1}{2} \gamma_{21} \|\dot{S}_i\|^{-1} + \frac{1}{2} \gamma_{22} \|\dot{\Delta}_i\|^{-1} \leq \|\dot{V}_i\| \leq \frac{1}{2} \gamma_{21} \|\dot{S}_i\|^{-1} + \frac{1}{2} \gamma_{22} \|\dot{\Delta}_i\|^{-1},
\]

(31)

where \( \gamma_{21} = V_{i,2}^{\phi} \gamma_{21} \) and \( \gamma_{22} = V_{i,2}^{\phi} \gamma_{21} \).

Proof: The proof is immediate when (27) and (30) are added.

The Theorem 2 represents that \( V_i \) can be manipulated by slipping and skidding variations whenever (31) is satisfied. Assuming that each extreme in (31) is a plane and by considering triples in the form \( \langle \|\dot{S}_i\|^{-1}, \|\dot{\Delta}_i\|^{-1}, \|\dot{V}_i\| \rangle \), we can say that the slipping and skidding variations must remain in these planes. The objective of the controller is to guarantee that these triples remain into the space confined by the planes (see Fig. 2):

\[
\pi_1: \|\dot{V}_i\| - \gamma_{21} \|\dot{S}_i\|^{-1} - \gamma_{22} \|\dot{\Delta}_i\|^{-1} = 0,
\]

\[
\pi_2: \|\dot{V}_i\| - \gamma_{21} \|\dot{S}_i\|^{-1} - \gamma_{22} \|\dot{\Delta}_i\|^{-1} = 0.
\]

(32)

4 Closed-loop analysis of curvilinear flexible law

By using classical results form static-feedback linearization (in D’Andréa-Novel et al., 1995) it can be observed that the system (5)-(6) can be generically transformed into a controllable linear system. In the work reported here, it will be included the modified auxiliary control law \( v \) such that

\[
\begin{cases}
\dot{y}_1 = y_2 \\
\dot{y}_2 = \left( \frac{\partial f(z)}{\partial z} + \frac{\partial g(z)}{\partial z} \right) \cdot Z_i^\phi(q) + \ldots + \Pi_3(z) v,
\end{cases}
\]

(33)

whenever \( \Pi_3(z) = \frac{\partial f(z)}{\partial z} \cdot S_i(q) \) and the linearizing vector function is given by (17), i.e., \( y_1 = f(z) \).

Thus, the change of coordinates can be defined by

\[
\left\{ \begin{array}{c}
y_1 = f(z) \\
y_2 = \Pi_3(z) \eta.
\end{array} \right.
\]

(35)

Consequently, the dynamic model defined by (11)-(13) can therefore be written under a standard form satisfying the Theorem 1. To this end, it will be made one more partition in the equation (12). Let this new partition described by

\[
\begin{cases}
\dot{z} = S_i(q) \eta + \varepsilon Z_i^\phi(q) \mu, \\
\dot{w}_1 = S_2(q) \eta + \varepsilon W_1(q) \mu + \varepsilon W_2(q) \mu, \\
\dot{w}_2 = W_3(q) \eta + \varepsilon W_4(q) \mu + W_5(q) \tau,
\end{cases}
\]

(37)

(38)

Now, let \( \gamma_{ref} = [y_1, y_2, y_2, y_2] \) be a bounded reference trajectory in a compact ball \( B_{ref}(0; \bar{r}_{ref}) \) such that \( y_{ref} \) is integrable. Thus, the tracking problem for model (37)-(40), when it is applied the auxiliary control law \( v \), can be reduced to the following lemma:

Lemma 3 With the change of coordinates defined by (35)-(36) when it is applied the state-feedback controller (14) with the auxiliary control law \( v \), the dynamical equations of

\[
\begin{cases}
\dot{y}_1 = \Pi_1(z) \Pi_1(z)^T \\
\dot{y}_2 = \Pi_1(z) \Pi_1(z)^T + \Pi_2(z) v,
\end{cases}
\]

(41)

have the following form:

\[
\begin{cases}
\dot{\gamma} = \Pi_1(z) Z_i^\phi(q) \eta + \varepsilon W_1(q) \mu + \varepsilon W_2(q) \mu
\end{cases}
\]

(42)

(43)

where \( \Pi_1(z) = \Pi_1(z) \Pi_1(z)^T \), such that

\[
\begin{align*}
i. \quad Z_i^\phi(q) &= \frac{\partial f(z)}{\partial z} \cdot Z_i^\phi(q) \eta + \varepsilon W_1(q) \mu, \\
&= \varepsilon W_1(q) \mu + W_2(q) \mu.
\end{align*}
\]

(44)

\[
\begin{align*}
ii. \quad G(y^*, w_1, \mu, \varepsilon, t) &= \Pi_3(z) \left( \frac{\partial f(z)}{\partial z} \cdot Z_i^\phi(q) \eta + \varepsilon W_1(q) \mu + \varepsilon W_2(q) \mu \right) \\
&+ \left( \frac{\partial g(z)}{\partial z} \cdot Z_i^\phi(q) \eta \right),
\end{align*}
\]

(45)

where \( H(q, \eta, \varepsilon, t) \) is given in (10).

\[
\begin{align*}
iii. \quad Z_i, S_i, G, S_2 \text{ are uniformly bounded with respect to } w_1.
\end{align*}
\]
Proof: Item i): With the change of coordinates defined by (35)-(36) and by using (37) then
\[
\dot{y}_1 = \frac{\partial f(z)}{\partial z} S_1(q) \eta + \frac{\partial f(z)}{\partial z} \varepsilon Z_1(q) \mu = y_2 + \frac{\partial f(z)}{\partial z} \varepsilon Z_1(q) \mu,
\]
By using the notation (36) then
\[
\dot{y}_2 = \Pi_1(z) \eta + \Pi_1(z) \dot{\eta},
\]
or, by using (39) which gives
\[
\dot{y}_2 = \Pi_1(z) \left( W_0 \dot{\eta} + W_1 \dot{x} + W_2 \dot{\mu} \right) + \left\{ \frac{\partial \Pi_1(z)}{\partial \eta} \eta \right\} + \Pi_1(z) \left( W_0 \dot{\eta} + W_1 \dot{x} + W_2 \dot{\mu} \right).
\]
Finally, with \(\tau(q, \eta, \tau)\) expressed as a function of \((y^*, \dot{y}^*, w_1, t)\) or \((y^*, \dot{y}^*, w_1, t)\), then \(G(q, \eta, \mu, 0, t) = G_0(q) \eta + G_2(q) H(q, \eta, \tau) + G_3(q) \tau + G_4(q, \eta) + G_5(q, \mu) \) and by using \(H(q, \eta, \tau)\), defined by (10), it is obtained:
\[
G(q, \eta, \mu, 0, t) = G_0(q) \eta + G_2(q) H(q, \eta, \tau) + G_3(q) \tau + G_4(q, \eta) + G_5(q, \mu) - G_0(q) \eta - G_2(q) H(q, \eta, \tau).
\]
Finally, with \(\tau(q, \eta, \tau)\) expressed as a function of \((y^*, w_1, t)\) or \((\dot{y}^*, \dot{w}_1, t)\), then \(G(q, \eta, \mu, 0, t)\) can be rewritten as follows:
\[
G(q, \eta, \mu, 0, t) = G_0(q) \eta + G_2(q)[\mu - H(q, \eta, \tau)] + G_3(q) \tau.
\]
Item iii): All quantities at this item depend on sine or cosine functions on \(\theta, \beta, \phi\) and \(\varphi\), which are angular variables and \(w_1 = [\beta_s, \varphi]^T\). Thus, the proof of item (iii) is immediate. This ends the proof.

Proposition 1 (Fernández et al., 2015) For a WMR that moves on a trajectory with slipping and skidding a necessary and sufficient condition for \(\|\dot{S}_i\| = 0\) and \(\|\Delta_t\| = 0\) is defined by
\[
\|R_s(q, \dot{q})\| = \|\dot{L}(q) + 2\lambda(q)\dot{q}\| = 0,
\]
such that
\[
\|A^T(q)A(q)\epsilon\mu\| \leq \|R_s(q, \dot{q})\|.
\]
In Proposition 1, \(A(q) = [D_s \Sigma(\beta_s) \theta_{N \times s}] S^+(q)\), where the matrices \(D_s\) and \(\Sigma(\beta)\) are known. For more details see (Campion et al., 1996).

Now, from (Dong, 2010; Fernández et al., 2014b), it can be said that when the Lyapunov function \(V = \frac{1}{2}(e_1^2 + e_2^2)\) is set then the following inequalities are valid:
\[
|\xi_1^* e_2 L_{e_2} e_2| \leq b^* - e_3\xi_2 \tan(h\xi_2^* e_2) + |e_3\xi_2^* e_2| \leq b^*,
\]
where \(h_1\) was set to 1. However, from Assumption 1, is also valid that
\[
\|A^T(q)A(q)e_m e_2 L_{e_2} e_2\| \leq \|e_1^* e_2 L_{e_2} e_2\| \leq b^* e_1^* e_2 L_{e_2} e_2 \leq b^* R_s(q, \dot{q})\|^{-1}
\]
Now, by using properly (46) in above inequality, it is obtained
\[
\|A^T(q)A(q)e m e_2 L_{e_2} e_2\| \leq \|e_1^* e_2 L_{e_2} e_2\| \leq b^* R_s(q, \dot{q})\|^{-1}
\]
Thus,
\[
V \leq -2\min\{k_2r_s, k_3r_s\} V + 2\delta V\|R_s(q, \dot{q})\|^{-1},
\]
\(e_1^* e_2 L_{e_2} e_2 \leq b^* R_s(q, \dot{q})\|^{-1}\).

Evaluating the regulable convergence radius
In order to assessing the controller proposed we will use the WMR shown in Fig. 1(b). Since that there are not centered wheels then in this WMR \(\delta_s = 0\). On the other hand, the vector \(q\) reduces to \(q = [x y \theta \psi_1 \varphi]^T\). The matrices associated with the kinematic and dynamic model are shown in (Fernández et al., 2014b). We assume that the reference trajectory is a rhombus with the diagonals equal to 6.28 m and each simulation lasts 4.5 s. The function \(h_s\) was set as in (45), where \(\Lambda(q)\) was restricted to \(\dim(Z_{ref}) = 3\) columns,
i.e.:
\[ \Lambda(q) = \begin{bmatrix} \frac{\sin \beta}{r_1 + r_2} & -\frac{\sin \beta}{r_1 + r_2} & -\frac{\cos \beta}{r_1 + r_2} & \frac{\cos \beta}{r_1 + r_2} & \frac{b \sin \beta}{r_1 + r_2} & \frac{b \cos \beta}{r_1 + r_2} \\ \frac{\sin \gamma}{r_3 + r_4} & -\frac{\sin \gamma}{r_3 + r_4} & -\frac{\cos \gamma}{r_3 + r_4} & \frac{\cos \gamma}{r_3 + r_4} & \frac{b \sin \gamma}{r_3 + r_4} & \frac{b \cos \gamma}{r_3 + r_4} \end{bmatrix} \]

where \( b \) is the displacement from each of driving wheels to the axis of symmetry of the WMR and \( r \) is the radius of each wheel. The numerical values of the parameters used in the simulations are the same presented in (Leroquais and D’Andrea-Novel, 1996), i.e., \( m = 1000 \text{ Kg}, L = 500 \text{ Kg-m}^2, I_w = 1.6 \text{ Kg-m}^2, L = 1 \text{ m}, b = 1 \text{ m} \) and \( r = 0.35 \text{ m} \). We choose \( k_2 = k_3 = 1 \) and \( R = 10^6 \text{ m} \) due to the rhombus has four corners (i.e., \( \text{curv}(s) \to \infty \)), thus \( R = 10^6 \text{ m} \) simulates a quasi-infinite curvature.

Tabela 1: Deviations of tracking by the WMR.

<table>
<thead>
<tr>
<th>( u_1^* ) [m/s]</th>
<th>d, with ( h_3 ) [cm]</th>
<th>d, without ( h_3 ) [cm]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.18</td>
<td>7.33 \times 10^{-2}</td>
<td>1.29</td>
</tr>
<tr>
<td>2.36</td>
<td>3.22</td>
<td>3.98</td>
</tr>
<tr>
<td>3.55</td>
<td>1.52</td>
<td>3.95</td>
</tr>
<tr>
<td>4.74</td>
<td>2.35</td>
<td>4.06</td>
</tr>
</tbody>
</table>

In Fig. 3 is shown the tracking made by the control law (14) with \( \varepsilon = 4 \times 10^{-11} \) in the cases with/without \( h_3 \), (label CC to represent case with \( h_3 \) and SC without \( h_3 \)). By using Assumption 2, Fig. 3 shows the tracking for four different velocities \( u_1^* = 1.18 \text{ m/s}, 2.36 \text{ m/s} \) (see Fig. 3(a)), \( 3.55 \text{ m/s} \) and \( 4.74 \text{ m/s} \) (see Fig. 3(b)). For each velocity was measured the deviation \( d \). A better detailing about the deviations is presented in Table 1.

In Fig. 4(b) is shown that triples \( \{\|\hat{S}_t\|^{-1}, \|\hat{\Delta}_t\|^{-1}, \|V_t\|\} \) remain into the space confined by the planes \( \pi_2: \|V_t\| - 5.3 \times 10^{-22} \|\hat{S}_t\|^{-1} - 3.6 \|\hat{\Delta}_t\|^{-1} = 0 \), \( \pi_1: \|V_t\| - 3.3 \times 10^{-22} \|\hat{S}_t\|^{-1} - 22.4 \|\hat{\Delta}_t\|^{-1} = 0 \), where \( 1.18 \leq \|V_t(q, \eta, \mu)\| \leq 4.74 \). However, when \( V_t = 61.25 \text{ m/s} \) the triple tends toward out of the confined space by the planes (see the red dot in Fig.4(a)).

6 Final remarks

The results shown in the previous section confirms the useful of regulable convergence radius for trajectory tracking problems in WMRs with flexibility when their motions at high velocities are subject to slipping and skidding. It can be seen that the tracking error shows a significant improving when the slipping and skidding variations subject to explicit conditions for the velocity at center of the wheels (represented by the confined space between planes \( \pi_1 \) and \( \pi_2 \)) are used to synthesize the auxiliary control law.

Referências


